Diploma Programme
Programme du diplôme
Programa del Diploma

# Mathematics: analysis and approaches
# Higher level
# Paper 3

20 May 2026

**Zone A** afternoon | **Zone B** afternoon | **Zone C** afternoon

1 hour 15 minutes

## Instructions to students

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

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8 pages

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

**1.** [Maximum mark: 30]

**This question considers different methods of summing powers of integers.**

The sum of the first $n$ positive integers, each raised to the power zero, is given by

$$S_0(n) = 1^0 + 2^0 + \ldots + n^0 \text{ where } n \in \mathbb{Z}^+.$$

(a) (i) Find the value of $S_0(4)$. [1]

(ii) Write down a simplified expression for $S_0(n)$. [1]

The sum of the first $n$ positive integers, each raised to the power $1$, is given by

$$S_1(n) = 1^1 + 2^1 + \ldots + n^1 \text{ where } n \in \mathbb{Z}^+.$$

(b) Use the fact that $S_1(n)$ is the sum of $n$ terms of an arithmetic sequence to show that $S_1(n) = \dfrac{n^2}{2} + \dfrac{n}{2}$. [3]

**(This question continues on the following page)**

**(Question 1 continued)**

The sum of the first $n$ square numbers is given by

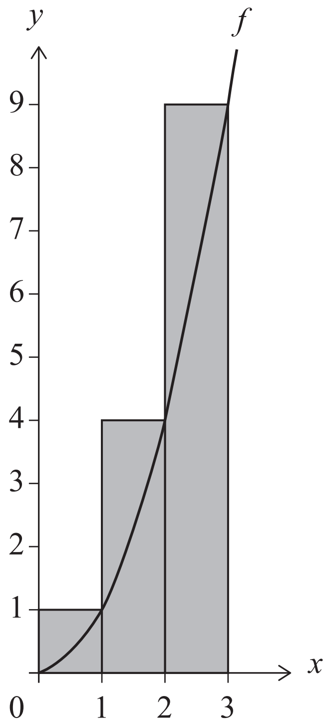
$$S_2(n) = 1^2 + 2^2 + 3^2 \ldots + n^2 \text{ where } n \in \mathbb{Z}^+.$$

(c) Find $S_2(5)$. [1]

(d) The following graph shows three rectangles, where the heights of the rectangles are consecutive square numbers, and the width of each rectangle is $1$.

The graph may be extended in a similar way to include any number of rectangles.

The curve $y = f(x)$ passes through the top right corner of each rectangle.




(i) Write down an expression for $f(x)$. [1]

(ii) By considering the total area of the first $n$ rectangles and the area under the curve, show that $S_2(n) > \frac{n^3}{3}$. [4]

(iii) Use a similar method to show that $S_2(n) < \frac{n^3}{3} + n^2 + n$. [4]

**(This question continues on the following page)**

**(Question 1 continued)**

Consider the sequence $u_n = n^3 - (n-1)^3$ for $n \in \mathbb{Z}^+$. Some of the values of $u_n$ and $\sum_{r=1}^{n} u_r$ are shown in the following table.

| $n$ | $u_n$ | $\sum_{r=1}^{n} u_r$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 7 | 8 |
| 3 | $a$ | $b$ |
| 4 | 37 | 64 |

(e) Find the value of

(i) $a$; [1]

(ii) $b$. [1]

(f) (i) Suggest an expression for $\sum_{r=1}^{n} u_r$ in terms of $n$. [1]

(ii) Use the definition of $u_n$ to prove that your suggestion is true. [3]

(g) (i) Find a quadratic expression for $u_n$. [2]

(ii) Hence, write $\sum_{r=1}^{n} u_r$ in terms of $S_2(n)$, $S_1(n)$ and $S_0(n)$. [2]

(iii) Hence, show that $S_2(n) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$. [3]

(iv) Show that the expression for $S_2(n)$ given in (g)(iii) is consistent with the inequalities in part (d). [2]

**2.** [Maximum mark: 25]

**This question asks you to explore curves in the complex plane defined by $r = a + b\sin\theta$, where $a$ and $b$ are real constants.**

A complex number $z$ can be represented by a point in the complex plane whose position is determined by the modulus, $r$, and the argument, $\theta$, of $z$.

The set of points which satisfy a given relationship between $r$ and $\theta$ form a curve in the complex plane.

Consider a family of curves defined by $r = a + b\sin\theta$, where $a$ and $b$ are real constants.

First consider the case where $a = 4$ and $b = 0$ so that $r = 4$.

The curve defined by $r = 4$ is the set of points in the complex plane which represent complex numbers with a modulus of $4$.

(a) Suppose that $z$ is a complex number such that $r = 4$.

Find all possible values of $z$ such that

(i) $z$ is purely real; [1]

(ii) $z$ is purely imaginary; [1]

(iii) the argument of $z$ is $\frac{\pi}{4}$. [1]

(b) Sketch the curve defined by $r = 4$ on an Argand diagram, showing clearly the values of the intercepts with the real and imaginary axes. [2]

Now consider the case where $a = 0$ and $b = 4$ so that $r = 4\sin\theta$, where $0 \le \theta \le \pi$.

(c) Suppose that $z$ is a complex number such that $r = 4\sin\theta$, where $0 \le \theta \le \pi$.

(i) Write down the greatest possible value of $r$. [1]

(ii) For this value of $r$, find $z$. [2]

(d) Use the identities $x \equiv r\cos\theta$ and $y \equiv r\sin\theta$ to show that the points on the curve defined by $r = 4\sin\theta$ lie on the curve $x^2 + (y-2)^2 = 4$, where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. [4]

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**(Question 2 continued)**

(e) Consider the curve $x^2 + (y-2)^2 = 4$.

(i) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $x$ and $y$. [3]

(ii) Hence or otherwise, find the values of $x$ for which the gradient of the curve is undefined. [2]

(f) The curve defined by $r = 4\sin\theta$, where $0 \le \theta \le \pi$, is a circle.

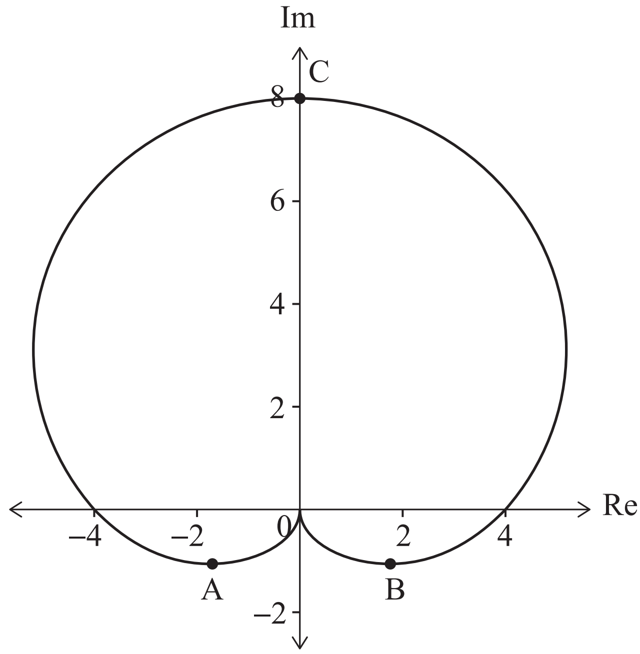
Sketch this curve on an Argand diagram, showing clearly the values of any intercepts with the real and imaginary axes. [1]

**(This question continues on the following page)**

**(Question 2 continued)**

Finally, consider the case where $a = 4$ and $b = 4$.

The curve defined by $r = 4 + 4\sin\theta$, where $0 \le \theta \le 2\pi$, is shown on the following Argand diagram. The curve has a horizontal tangent at point $\mathrm{A}$, point $\mathrm{B}$ and point $\mathrm{C}$. The real parts of the numbers represented by $\mathrm{A}$, $\mathrm{B}$ and $\mathrm{C}$ are negative, positive and zero, respectively.




(g) Use the identities $x \equiv r\cos\theta$ and $y \equiv r\sin\theta$ to show that the gradient of the curve at the point $(r\cos\theta, r\sin\theta)$ is given by $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\cos\theta(1 + 2\sin\theta)}{1 - \sin\theta - 2\sin^2\theta}$. [5]

(h) Hence, find the value of $\theta$ which corresponds to the point $\mathrm{A}$. [2]